GROMOV HYPERBOLICITY OF PERIODIC PLANAR GRAPHS

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Abstract. The study of hyperbolic graphs is an interesting topic since the hyperbolicity of a geodesic metric space is equivalent to the hyperbolicity of a graph related to it. The main result in this paper is a very simple characterization of the hyperbolicity of a large class of periodic planar graphs.

Keywords: Planar Graphs; Periodic Graphs; Gromov Hyperbolicity; Infinite Graphs; Geodesics.

AMS Subject Classification numbers 2010: 05C10; 05C63; 05C75; 05A20.

1. Introduction.

Hyperbolic spaces play an important role in geometric group theory and in the geometry of negatively curved spaces (see [1, 2, 3]). The concept of Gromov hyperbolicity grasps the essence of both negatively curved spaces like the classical hyperbolic space or Riemannian manifolds of negative sectional curvature, and of discrete spaces like trees and Cayley graphs of many finitely generated groups. It is remarkable how a simple concept leads to such a rich general theory (see [1, 2, 3]).

The study of mathematical properties of Gromov hyperbolic spaces and its applications is a topic of recent and increasing interest in graph theory; see, for instance [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18].

The theory of Gromov spaces was used initially for the study of finitely generated groups (see [3] and the references therein), where it was observed to have a practical importance. This theory was applied mainly to the study of automatic groups (see [19]), which play a role in computation science. The concept of hyperbolicity appears also in discrete mathematics, algorithms and networking. For example, it has been shown empirically in [20] that the internet topology embeds with better accuracy into hyperbolic space than into Euclidean space of comparable dimension. A few algorithmic problems in hyperbolic spaces and hyperbolic graphs have been considered in recent papers (see [21, 22, 23, 24]). Another interesting application of these spaces is secure transmission of information on the internet (see [9, 10, 11]). In particular, the hyperbolicity plays a key role in the spread of viruses through the network (see [10, 11]). The hyperbolicity is also useful in the study of DNA data (see [6]).

In [25, Section 1.3] is observed that the hyperbolicity of a geodesic metric space is equivalent to the hyperbolicity of a graph related to it (see also [14, 16, 17]). Hence, establishing hyperbolicity criteria for graphs will be of interest to us.

Let us state some basic facts about Gromov's spaces. If $\gamma : [a, b] \longrightarrow X$ is a continuous curve in a metric space (X, d), we say that γ is a *geodesic* if it is an isometry, *i.e.*, $L(\gamma|_{[s,t]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$. The space X is a *geodesic metric space* if for every $x, y \in X$ there exists a geodesic joining x

and y; denote by $[xy]$ any of such geodesics (since uniqueness of geodesics is not required, this notation is ambiguous, but it is convenient). It is clear that every geodesic metric space is path-connected.

By a graph G we mean a set of points called vertices connected by (undirected) edges; the set of vertices is denoted by $V(G)$ and the set of edges by $E(G)$; we assume also that each edge has assigned a length. In order to consider a graph G as a geodesic metric space, identify (by an isometry) any edge $[u,v] \in E(G)$ with the real interval $[0,l]$ (if $l := L([u,v])$); therefore, any point in the interior of an edge is a point of G. Then G is naturally equipped with a distance defined on its points, induced by taking shortest paths.

If X is a geodesic metric space and $J = \{J_1, J_2, \dots, J_n\}$ is a polygon, with sides $J_j \subseteq X$, the polygon J is δ-*thin* if for every $x \in J_i$ one has that $d(x, \cup_{j\neq i} J_j) \le \delta$. Denote by $\delta(J)$ the sharp thin constant of J, *i.e.*, $\delta(J) := \inf\{\delta : J \text{ is } \delta\text{-thin}\}$. If $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$. The space X is δ-*hyperbolic* if every geodesic triangle in X is δ-thin. Let us denote be $\delta(X)$ the sharp hyperbolicity constant of X, *i.e.*, $\delta(X) := \sup\{\delta(T) : T \text{ is a geodesic triangle in } X\}$. The space X is *hyperbolic* if X is δ-hyperbolic for some δ. Note that if X is δ-hyperbolic, then every geodesic polygon with n sides is $(n-2)\delta$-thin; in particular, every geodesic quadrilateral is 2δ-thin.

As a remark, the main examples of hyperbolic graphs are trees. In fact, the hyperbolicity constant of a geodesic metric space can be viewed as a measure of how "tree-like" the space is, since those spaces X with $\delta(X) = 0$ are precisely the metric trees. This is an interesting subject since, in many applications, one finds that the borderline between tractable and intractable cases may be the tree-like degree of the structure to be dealt with (see, e.g., [26]).

It is worth pointing out that deciding whether or not a space is hyperbolic is usually extraordinarily difficult: Note that, first of all, one needs to consider an arbitrary geodesic triangle T, and calculate the minimum distance from an arbitrary point P of T to the union of the other two sides of the triangle to which P does not belong to. And then, to take the supremum over all the possible choices for P and then over all the possible choices for T. Without disregarding the difficulty of solving this minimax problem, notice that in general the main obstacle is that the location of geodesics in the space is not usually known.

One of the main questions in the study of any mathematical property is to characterize it in a simple way. If the property is very difficult to characterize (as in the case of hyperbolicity), a natural strategy is to do so for a subclass of objects. In this paper a very simple characterization of the hyperbolicity of periodic tessellation graphs of $\mathbb{R}^2$ is given (see Theorem 4.1 and Definitions 3.2 and 3.3).

Theorem 4.1 characterizes the hyperbolicity of any periodic tessellation graph G in terms of the hyperbolicity of a "period graph" G^* (G can be obtained by pasting infinitely many copies of G^*, see Definition 3.3). As a first intuition, one might think that the hyperbolicity of G^* guarantees the hyperbolicity of G; however, this does not hold for the Cayley graph G of $\mathbb{Z} \times \mathbb{Z}$, where one can take as G^* the Cayley graph of $\mathbb{Z} \times \mathbb{Z}_2$ (note that G^* is hyperbolic and G is not hyperbolic). Theorem 4.1 states that G is hyperbolic if and only if G^* is hyperbolic and the geodesic lines bordering G^* "diverge" (this last condition is not satisfied in $\mathbb{Z} \times \mathbb{Z}_2$).

The outline of the paper is as follows. In Section 2 the needed background is collected. Section 3 contains the technical results used in the proof of the main theorem, which appears in Section 4.

2. Background on Gromov hyperbolic spaces.

Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \longrightarrow Y$ is said to be an (α, β)-*quasi-isometric embedding*, with constants $\alpha \ge 1$, $\beta \ge 0$ if, for every $x, y \in X$:

$$\alpha^{-1} d_X(x,y) - \beta \le d_Y(f(x), f(y)) \le \alpha d_X(x,y) + \beta.$$

The function f is ε-*full* if for each $y \in Y$ there exists $x \in X$ with $d_Y(f(x), y) \le \varepsilon$.

A map $f : X \longrightarrow Y$ is said to be a *quasi-isometry*, if there exist constants $\alpha \ge 1$, $\beta, \varepsilon \ge 0$ such that f is a ε-full (α, β)-quasi-isometric embedding.

Two metric spaces X and Y are *quasi-isometric* if there exists a quasi-isometry $f : X \longrightarrow Y$.

An (α,β)-*quasigeodesic* of a metric space X is an (α,β)-quasi-isometric embedding $\gamma : I \longrightarrow X$, where I is an interval of $\mathbb{R}$. A *quasigeodesic* is an (α,β)-quasigeodesic for some $\alpha \geq 1,\ \beta \geq 0$. Note that a $(1,0)$-quasigeodesic is a geodesic. A *geodesic line* is a geodesic with domain $\mathbb{R}$. A *geodesic ray* is a geodesic with domain $[0,\infty)$.

Let X be a metric space, Y a non-empty subset of X and ε a positive number. The ε-*neighborhood* of Y in X, denoted by $\mathcal{V}_\varepsilon(Y)$ is the set $\{x \in X : d_X(x,Y) \leq \varepsilon\}$.

The *Hausdorff distance* between two non-empty subsets Y and Z of X, denoted by $\mathcal{H}(Y,Z)$, is the number defined by:

$$\inf\{\varepsilon > 0 : Y \subset \mathcal{V}_\varepsilon(Z) \ and \ Z \subset \mathcal{V}_\varepsilon(Y)\}.$$

Two of the fundamental properties of hyperbolic spaces are the following:

Theorem 2.1 (Invariance of hyperbolicity)**.** *Let $f : X \longrightarrow Y$ be an (α,β)-quasi-isometric embedding between the geodesic metric spaces X and Y. If Y is hyperbolic, then X is hyperbolic.*

Besides, if f is ε-full for some $\varepsilon \geq 0$ (a quasi-isometry), then X is hyperbolic if and only if Y is hyperbolic.

Theorem 2.2 (Geodesic stability)**.** *For given constants $\alpha \geq 1$ and $\beta,\delta \geq 0$ there exists a constant $H = H(\delta,\alpha,\beta)$ such that for every δ-hyperbolic geodesic metric space and for every pair of (α,β)-quasigeodesics g,h with the same endpoints, $\mathcal{H}(g,h) \leq H$.*

If X is a metric space, define the *Gromov product* of $x,y \in X$ with base point $w \in X$ by

$$(x,y)_w := \frac{1}{2}\big(d(x,w) + d(y,w) - d(x,y)\big).$$

If X is a Gromov hyperbolic space, it holds

$$(x,z)_w \geq \min\big\{(x,y)_w,(y,z)_w\big\} - \delta \tag{2.1}$$

for every $x,y,z,w \in X$ and some constant $\delta \geq 0$ (see, *e.g.*, [1, Proposition 2.1] or [2, p.41]). Let us denote by $\delta^*(X)$ the sharp constant for this inequality, *i.e.*,

$$\delta^*(X) := \sup\big\{\min\big\{(x,y)_w,(y,z)_w\big\} - (x,z)_w : x,y,z,w \in X\big\}.$$

Remark 2.3. *If X is a geodesic metric space, it is known that* (2.1) *is, in fact, equivalent to our definition of Gromov hyperbolicity; furthermore, $\delta^*(X) \leq 4\,\delta(X)$ and $\delta(X) \leq 3\,\delta^*(X)$ (see,* e.g., [1, Proposition 2.1] *or* [2, p.41]*).*

If D is a closed subset of X, always consider in D the *inner metric* obtained by the restriction of the metric in X, that is

$$d_D(z,w) := \inf\big\{L_X(\gamma) : \gamma \subset D \text{ is a continuous curve joining } z \text{ and } w\big\} \geq d_X(z,w)\,.$$

Consequently, $L_D(\gamma) = L_X(\gamma)$ for every curve $\gamma \subset D$.

In an informal way, a tessellation, T, on $\mathbb{R}^2$ is a partition of $\mathbb{R}^2$ by geometric shapes (called tiles) with no overlaps and no gaps. The tessellation graph associated to T is the union of the boundaries of the tiles. More precisely, for $n \geq 1$, an n-cell is a topological space homeomorphic to the open ball in $\mathbb{R}^n$. A 0-cell is a singleton space. A *tesselation* on $\mathbb{R}^2$ is a CW 2-complex on $\mathbb{R}^2$ such that every point on $\mathbb{R}^2$ is contained in some n-cell of the complex for some $n \in \{0,1,2\}$. A *tessellation graph* is the 1-skeleton (the set of 0-cells and 1-cells). The edges (1-cells) of a tessellation graph are just rectifiable paths (paths with finite Euclidean length) in $\mathbb{R}^2$ and have the length induced by the Euclidean metric. Note that this class of graphs contains as particular cases many planar graphs.

3. Technical lemmas.

Since the proof of our main result (Theorem 4.1) is long and technical, in order to make the arguments more transparent, we collect some results needed along the proof in technical lemmas. Let us start with the definition of periodic graph.

Definition 3.1. *Let G be a tessellation graph of $\mathbb{R}^2$. Then, G is* periodic *if there exists $(u,v) \in \mathbb{R}^2 \setminus \{(0,0)\}$ such that $T(G)=G$, where $T:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is defined as $T(x,y)=(x,y)+(u,v)$. In this case, T is a* periodic transformation *of G.*

Definition 3.2. *A periodic tessellation graph G of $\mathbb{R}^2$ is* normalized *if $T(x,y)=(x+k_0,y)$ is a periodic transformation of G for some positive constant $k_0>0$, and $\sigma := \mathbb{R}\times\{0\}$ is contained in G.*

Note that in order to study the hyperbolicity of a periodic tessellation graph G of $\mathbb{R}^2$, by applying a rotation and/or a lift, without loss of generality one can assume that $T(x,y)=(x+k_0,y)$ with $k_0>0$. Furthermore, one can assume that $\sigma := \mathbb{R}\times\{0\} \subset G$, since otherwise the tessellation graph obtained by adding σ to G is quasi-isometric to G. Hence, in order to study the hyperbolicity, assume that every periodic tessellation graph of $\mathbb{R}^2$ is normalized.

Definition 3.3. *Denote by U (respectively, L) the closed upper (respectively, lower) half-plane in $\mathbb{R}^2$. If G is a normalized periodic tessellation graph of $\mathbb{R}^2$, then γ is a* fundamental ray *of G in U (respectively, in L) if it is a geodesic ray of G in U (respectively, in L) starting in $(0,0)$ verifying the following property: if B is the closed connected set in U (respectively, in L) bounded by γ and $T(\gamma)$, then $\cup_{n\in\mathbb{Z}}T^n(B)=U$ (respectively, $\cup_{n\in\mathbb{Z}}T^n(B)=L$). In this case, γ_0 is said a* fundamental line *of G if $\gamma_0=\gamma_1\cup\gamma_2$, where γ_1 is a fundamental ray of G in U and γ_2 is a fundamental ray of G in L; if B is the closed connected set in $\mathbb{R}^2$ bounded by γ_0 and $T(\gamma_0)$, then $\cup_{n\in\mathbb{Z}}T^n(B)=\mathbb{R}^2$. The* period graph *$G^*$ of G (with respect to γ_0) is the subgraph $G^* := G\cap B$; then $\cup_{n\in\mathbb{Z}}T^n(G^*)=G$.*

The following result is a main tool in order to state our main result.

Lemma 3.4. *For any normalized periodic tessellation graph G of $\mathbb{R}^2$ there exists a fundamental line.*

Proof. By symmetry, it suffices to show that there exists a fundamental ray of the graph G in U.

Given $s>0$, let E_s be the set of geodesics starting in $p=(0,0)$ and finishing in some point q with $d_G(p,q)\le s$; hence, any geodesic in E_s is contained in the closed ball $\overline{B_G(p,s)}$ (if a geodesic in E_s has length $s'<s$, and it can be considered as a map defined on $[0,s]$ which is constant in the interval $[s',s]$). Let us consider in E_s the uniform convergence topology. Since the closed ball $\overline{B_G(p,s)}$ is compact, E_s is compact by Arzelà-Ascoli's Theorem. If $\gamma : J\to G$ is a geodesic with $J=[0,a]$, define $J^s := J\cap[0,s]$ and denote by $\gamma^{(s)}$ the restriction of γ to J^s (hence, $\gamma^{(s)}\in E_s$).

Note that if a geodesic starts in $p=(0,0)$ and exits from σ, then it does not return to σ. For each natural number $n>s$, choose a point r_n in the open upper half-plane with $d_G(r_n,\sigma)\ge n$ and a geodesic $[pr_n]$. The geodesic $[pr_n]$ exits from σ in the point p_n; let u be such that $T^{-u}(p_n)\in[pr_n]$ and $d_G(T^{-u}(p_n),p)\le s$. Then $\gamma_n' := [pT^{-u}(p_n)]\cup T^{-u}([p_nr_n])$ is a geodesic; note that $L(\gamma_n'\cap\sigma)\le s$ and $L(\gamma_n')\ge n$. Iterating this argument one can obtain a geodesic γ_n with the following property: for every horizontal line σ' one gets $L(\gamma_n\cap\sigma')\le s$ and $L(\gamma_n)\ge n$.

Since E_s is compact, for each $s>0$ there exists a subsequence $\{\gamma_{s,m}\}_m$ from $\{\gamma_n\}_n$ such that $\{\gamma_{s,m}^{(s)}\}_m$ converges uniformly. Cantor's diagonal argument gives a subsequence $\{g_n\}_n$ from $\{\gamma_n\}_n$ such that the sequence $\{g_n^{(s)}\}_n$ converges uniformly to a geodesic g^s; since $(g^s)^{(s')}=g^{s'}$ if $s'<s$, these geodesics g^s define a geodesic ray g starting in p. One can check that g is contained in U and that $L(g\cap\sigma')\le s$ for every horizontal line σ'. If $g=(u,v)$, let us check that $\limsup_{t\to\infty}v(t)=\infty$: since g does not contain an horizontal ray, if $v(t)\le M$ for some constant M and every $t\ge0$, then $\{T^{-n}(g)\}_{n\ge0}$ accumulates, which is a contradiction.

Let us denote by B the closed connected set in U bounded by g and $T(g)$; since $\limsup_{t\to\infty}v(t)=\infty$, one obtains $\cup_{n\in\mathbb{Z}}T^n(B)=U$. □

Lemma 3.5. *Let G be a δ-hyperbolic graph and let γ_0 be either a geodesic line or a geodesic ray in G. For any $x\in G$, denote by x' a point in γ_0 with $d_G(x,\gamma_0)=d_G(x,x')$. Then, for any $w\in\gamma_0$,*

$$d_G(x,x')+d_G(x',w)-8\delta \le d_G(x,w)\le d_G(x,x')+d_G(x',w).$$

Proof. The upper bound is just the triangle inequality; let us prove the lower bound. Given any geodesic triangle $T=\{x,x',w\}$, let $a\in[x'w], b\in[xw], c\in[xx']$ be the points verifying $d_G(x,b)=d_G(x,c)$, $d_G(x',a)=d_G(x',c)$, $d_G(w,a)=d_G(w,b)$. Since G is δ-hyperbolic, it is well known that $d_G(a,b),d_G(a,c),d_G(b,c)\le 4\delta$ (see, *e.g.*, [1, Proposition 2.1]). Since $d_G(x,\gamma_0)=d_G(x,x')$ and $a\in\gamma_0$, it follows $d_G(x',a)=d_G(c,x')\le d_G(c,a)\le 4\delta$ and one deduces $d_G(x,w)=d_G(x,b)+d_G(b,w)=d_G(x,c)+d_G(a,w)\ge d_G(x,c)+d_G(c,x')-4\delta+d_G(x',a)-4\delta+d_G(a,w)=d_G(x,x')+d_G(x',w)-8\delta$. $\square$

A subgraph Γ of G is *isometric* if $d_\Gamma(x,y)=d_G(x,y)$ for every $x,y\in\Gamma$. The following result appears in [15, Lemma 5].

Lemma 3.6. *If Γ is an isometric subgraph of G, then $\delta(\Gamma)\le\delta(G)$.*

Lemma 3.7. *Let G be a graph and let γ_0 be either a geodesic line or a geodesic ray in G such that γ_0 disconnects G. Let G_1',G_2' be two connected components of $G\setminus\gamma_0$ and define $G_1:=G_1'\cup\gamma_0$ and $G_2:=G_2'\cup\gamma_0$. Then G_1,G_2 are isometric subgraphs of G. For any $x\in G$, denote by x' a point in γ_0 with $d_G(x,\gamma_0)=d_G(x,x')$. If G_1,G_2 are δ-hyperbolic and $x_1\in G_1,x_2\in G_2$, then*

$$d_G(x_1,x_1')+d_G(x_1',x_2')+d_G(x_2',x_2)-16\delta\le d_G(x_1,x_2)\le d_G(x_1,x_1')+d_G(x_1',x_2')+d_G(x_2',x_2).$$

Proof. First of all, it will be shown that G_1,G_2 are isometric subgraphs of G. The inequality $d_G(x,y)\le d_{G_i}(x,y)$ for every $x,y\in G_i$ is direct. In order to prove the reverse inequality, let us fix $x,y\in G_i$ and a geodesic σ in G joining them $\sigma:[0,l]\to G$. If σ is contained in G_i, then $d_G(x,y)=d_{G_i}(x,y)$. Otherwise, σ intersects γ_0. If σ' is a subcurve of σ joining two points u,v in γ_0 and g is the subcurve of γ_0 joining u,v, then $L(\sigma')=L(g)$ since γ_0 is a geodesic. Consequently, $d_G(x,y)\ge d_{G_i}(x,y)$ and conclude $d_{G_i}(x,y)=d_G(x,y)$ for every $x,y\in G_i$.

Let w be a point in $\gamma_0\cap[x_1x_2]$. Since G_1,G_2 are δ-hyperbolic, Lemma 3.5 implies

$$\begin{aligned}
d_G(x_1,x_2)&=d_{G_1}(x_1,w)+d_{G_2}(w,x_2)\\
&\ge d_{G_1}(x_1,x_1')+d_{G_1}(x_1',w)-8\delta+d_{G_2}(w,x_2')+d_{G_2}(x_2',x_2)-8\delta\\
&=d_G(x_1,x_1')+d_G(x_1',w)+d_G(w,x_2')+d_G(x_2',x_2)-16\delta\\
&\ge d_G(x_1,x_1')+d_G(x_1',x_2')+d_G(x_2',x_2)-16\delta.
\end{aligned}$$

$\square$

Lemma 3.8. *Let G be a graph and let γ_0 be either a geodesic line or a geodesic ray in G such that γ_0 disconnects G. Let G_1',G_2' be two connected components of $G\setminus\gamma_0$ and define $G_1:=G_1'\cup\gamma_0$ and $G_2:=G_2'\cup\gamma_0$. For any $x\in G$, denote by x' a point in γ_0 with $d_G(x,\gamma_0)=d_G(x,x')$. If G_1,G_2 are δ-hyperbolic, $x_1\in G_1,x_2\in G_2$ and $w\in\gamma_0$, then*

$$d_G(x_2',w)-d_G(x_1,x_2')-16\delta\le d_G(x_2,w)-d_G(x_1,x_2)\le d_G(x_2',w)-d_G(x_1,x_2')+16\delta.$$

Proof. Since γ_0 is a geodesic line, G_1,G_2 are isometric subgraphs in G. Then Lemma 3.5 (applied to G_1 and G_2) and Lemma 3.7 imply

$$\begin{aligned}
d_G(x_2,w)-d_G(x_1,x_2)&\ge d_G(x_2,x_2')+d_G(x_2',w)-8\delta-d_G(x_1,x_1')-d_G(x_1',x_2')-d_G(x_2',x_2)\\
&\ge d_G(x_2',w)-d_G(x_1,x_2')-16\delta,\\
d_G(x_2,w)-d_G(x_1,x_2)&\le d_G(x_2,x_2')+d_G(x_2',w)-d_G(x_1,x_1')-d_G(x_1',x_2')-d_G(x_2',x_2)+16\delta\\
&\le d_G(x_2',w)-d_G(x_1,x_2')+16\delta.
\end{aligned}$$

$\square$

The following result appears in [3, Corollary 1.1B] and [1, Proposition 2.2].

Lemma 3.9. *Let X be a metric space verifying for some fixed $w_0\in X$*

$$(x,z)_{w_0}\ge\min\left\{(x,y)_{w_0},(y,z)_{w_0}\right\}-\delta$$

for every $x,y,z\in X$ and some constant $\delta\ge 0$, then (2.1) holds with constant 2δ for every $x,y,z,w\in G$.

Lemma 3.10. *Let G be a graph and let γ_0 be either a geodesic line or a geodesic ray in G such that $G \setminus \gamma_0$ has two connected components G_1', G_2'. Define $G_1 := G_1' \cup \gamma_0$ and $G_2 := G_2' \cup \gamma_0$. If G is δ-hyperbolic, then G_1, G_2 are δ-hyperbolic. If G_1, G_2 are δ-hyperbolic, then G is 120δ-hyperbolic.*

Proof. Note that Lemma 3.7 gives that G_1, G_2 are isometric subgraphs of G. Therefore, if G is δ-hyperbolic, then G_1, G_2 are δ-hyperbolic by Lemma 3.6.

Assume now that G_1, G_2 are δ-hyperbolic. We will prove that G is 120δ-hyperbolic by using Remark 2.3 and Lemmas 3.8 and 3.9.

Let us fix $w \in \gamma_0$ and $x, y, z \in G$. Without loss of generality one can assume either that $x, y, z \in G_1$, or $x, y \in G_1$ and $z \in G_2$, or $x, z \in G_1$ and $y \in G_2$ (observe that in our argument x and z play a symmetric role, but y play another role since it appears in a different place in the inequalities).

If $x, y, z \in G_1$, since G_1 is δ-hyperbolic, then Remark 2.3 gives

$$(x, z)_w \ge \min\big\{(x, y)_w, (y, z)_w\big\} - 4\delta.$$

If $x, y \in G_1$ and $z \in G_2$, then Lemma 3.8 gives $d_G(z, w) - d_G(x, z) \ge d_G(z', w) - d_G(x, z') - 16\delta$ and $d_G(z', w) - d_G(y, z') \ge d_G(z, w) - d_G(y, z) - 16\delta$. Since G_1 satisfies (2.1) with constant 4δ, deducing

$$\begin{aligned}
2(x, z)_w &= d_G(x, w) + d_G(z, w) - d_G(x, z) \ge d_G(x, w) + d_G(z', w) - d_G(x, z') - 16\delta \\
&= 2(x, z')_w - 16\delta \ge \min\big\{2(x, y)_w, 2(y, z')_w\big\} - 8\delta - 16\delta \\
&= \min\big\{d_G(x, w) + d_G(y, w) - d_G(x, y), d_G(y, w) + d_G(z', w) - d_G(y, z')\big\} - 24\delta \\
&\ge \min\big\{d_G(x, w) + d_G(y, w) - d_G(x, y), d_G(y, w) + d_G(z, w) - d_G(y, z)\big\} - 16\delta - 24\delta \\
&= 2\min\big\{(x, y)_w, (y, z)_w\big\} - 40\delta.
\end{aligned}$$

If $x, z \in G_1$ and $y \in G_2$, then Lemma 3.8 gives $d_G(y', w) - d_G(x, y') \ge d_G(y, w) - d_G(x, y) - 16\delta$ and $d_G(y', w) - d_G(y', z) \ge d_G(y, w) - d_G(y, z) - 16\delta$. Since G_1 satisfies (2.1) with constant 4δ, one concludes

$$\begin{aligned}
2(x, z)_w &\ge \min\big\{2(x, y')_w, 2(y', z)_w\big\} - 8\delta \\
&= \min\big\{d_G(x, w) + d_G(y', w) - d_G(x, y'), d_G(y', w) + d_G(z, w) - d_G(y', z)\big\} - 8\delta \\
&\ge \min\big\{d_G(x, w) + d_G(y, w) - d_G(x, y), d_G(y, w) + d_G(z, w) - d_G(y, z)\big\} - 24\delta \\
&= 2\min\big\{(x, y)_w, (y, z)_w\big\} - 24\delta.
\end{aligned}$$

Hence, in any case, $(x, z)_w \ge \min\big\{(x, y)_w, (y, z)_w\big\} - 20\delta$, if $w \in \gamma_0$. Consequently, Lemma 3.9 gives that (2.1) holds with constant 40δ, and Remark 2.3 gives that G is 120δ-hyperbolic. $\square$

Lemma 3.11. *Let G be a normalized periodic tessellation graph of $\mathbb{R}^2$ such that G^* is δ^*-hyperbolic and $\lim_{|z|\to\infty, z\in\gamma_0} d_G(z, T(z)) = \infty$ for some choice of the fundamental line γ_0. Assume also that γ_0 is a geodesic line. Let $x \in T^j(G^*)$ and $y \in T^k(G^*)$ with $j \le k$. Then there exists constants M, N, which just depend on G^* and δ^*, verifying the following properties:*

(1) *For each geodesic γ joining x and y there exists another geodesic γ' joining x and y, with γ' contained in $\cup_{i=j}^k T^i(G^*)$ and such that $\mathcal{H}(\gamma, \gamma') \le M$. Furthermore, $\gamma' \cap T^i(G^*)$ is a connected set for each $j \le i \le k$.*

(2) *If $j + 2 \le k$, then for each $j < i < k$ there exists a point $z_i \in \gamma'$ with $d_{T^i(G^*)}(z_i, \sigma \cap T^i(G^*)) \le N$.*

Remark 3.12. *The proof of Lemma 3.11 gives that the same result holds for periodic tessellation graphs of U or L.*

Proof of Lemma 3.11. Note that G^* is an isometric subgraph, since γ_0 (and $T(\gamma_0)$) is a geodesic line. Therefore, if $x, y \in G^*$ and $A, B \subset G^*$, then $d_{G^*}(x, y) = d_G(x, y)$ and $\mathcal{H}_{G^*}(A, B) = \mathcal{H}(A, B)$.

In order to prove (1), first of all we are going to prove that there exists a constant M, which just depends on G^* and δ^*, with the following property: if $u, v \in \gamma_0$, η_0 is the subset of γ_0 joining u and v, and η is any geodesic joining u and v, then $\mathcal{H}(\eta_0, \eta) \le M$.

Let H be the constant $H = H(\delta^*, 1, 0)$ in Theorem 2.2. One can check that if $\lim_{|z|\to\infty, z\in\gamma_0} d_G(z, T(z)) = \infty$, then $\lim_{|z|\to\infty, z\in\gamma_0} d_G(z, T(\gamma_0)) = \infty$ and $\lim_{|z|\to\infty, z\in\gamma_0} d_G(z, T^{-1}(\gamma_0)) = \infty$. Then there exists a constant R such that if $z \in \gamma_0$ and $d_{G^*}(z, \sigma \cap G^*) > R$, then $d_{G^*}(z, T(\gamma_0)) > H$ and $d_{T^{-1}(G^*)}(z, T^{-1}(\gamma_0)) > H$.

Assume first that η is contained either in G^* or in $T^{-1}(G^*)$. Without loss of generality, assume that η is contained in G^*. Then $\{\eta_0, \eta\}$ is a geodesic bigon in G^*, and Theorem 2.2 gives $\mathcal{H}(\eta_0, \eta) \le H$. If η is contained in $G^* \cup T^{-1}(G^*)$, apply the previous argument to each subset of η contained either in G^* or in $T^{-1}(G^*)$, obtaining also $\mathcal{H}(\eta_0, \eta) \le H$.

Assume now that η is not contained in $G^* \cup T^{-1}(G^*)$ and that it is contained either in $\cup_{i<0} T^i(G^*)$ or in $\cup_{i\ge 0} T^i(G^*)$. Without loss of generality one can assume that η is contained in $\cup_{i\ge 0} T^i(G^*)$. If $\eta : [0, l] \to \cup_{i\ge 0} T^i(G^*)$, then define

$$a := \max\{s \in [0, l] / \, \eta(t) \in G^* \ \forall t \in [0, s]\}, \qquad b := \min\{s \in [0, l] / \, \eta(t) \in G^* \ \forall t \in [s, l]\}.$$

If η_1 is the subset of $T(\gamma_0)$ joining $\eta(a)$ and $\eta(b)$, then define $\eta_2 := \eta([0, a]) \cup \eta_1 \cup \eta([b, l])$. Hence, $\{\eta_0, \eta_2\}$ is a geodesic bigon in G^*, and Theorem 2.2 gives $\mathcal{H}(\eta_0, \eta_2) \le H$. Since $d_G(\eta_0, \eta(a)), d_G(\eta_0, \eta(b)) \le H$ and $\eta(a), \eta(b) \in T(\gamma_0)$, one has $d_{G^*}(\eta(a), \sigma \cap G^*), d_{G^*}(\eta(b), \sigma \cap G^*) \le R$; thus, conclude $d_{G^*}(\eta(a), \eta(b)) \le 2R + L(\sigma \cap G^*)$. Therefore, $\mathcal{H}(\eta_1, \eta|_{[a,b]}) \le R + L(\sigma \cap G^*)/2$ and $\mathcal{H}(\eta_0, \eta) \le M := H + R + L(\sigma \cap G^*)/2$.

In the general case, one can apply the previous argument to each subset of η contained either in $\cup_{i<0} T^i(G^*)$ or in $\cup_{i\ge 0} T^i(G^*)$, to obtain also $\mathcal{H}(\eta_0, \eta) \le M$.

Assume now that γ is contained either in $\cup_{i\ge j} T^i(G^*)$ or in $\cup_{i\le k} T^i(G^*)$. Without loss of generality one can assume that γ is contained in $\cup_{i\ge j} T^i(G^*)$. If $\gamma : [0, L] \to \cup_{i\ge j} T^i(G^*)$, then define

$$a := \max\{s \in [0, L] / \, \eta(t) \in \cup_{i=j}^k T^i(G^*) \ \forall t \in [0, s]\}, \qquad b := \min\{s \in [0, L] / \, \eta(t) \in \cup_{i=j}^k T^i(G^*) \ \forall t \in [s, L]\}.$$

If γ_1 is the subset of $T^{k+1}(\gamma_0)$ joining $\gamma(a)$ and $\gamma(b)$, then define $\gamma' := \gamma([0, a]) \cup \gamma_1 \cup \gamma([b, L]) \subset \cup_{i=j}^k T^i(G^*)$. But $\mathcal{H}(\gamma_1, \gamma|_{[a,b]}) \le M$ and, hence, $\mathcal{H}(\gamma, \gamma') \le M$.

In the general case, apply the previous argument to each subset of γ contained either in $\cup_{i>k} T^i(G^*)$ or in $\cup_{i<j} T^i(G^*)$, therefore $\mathcal{H}(\gamma, \gamma') \le M$.

Applying again the previous argument, $\gamma' \cap T^i(G^*)$ is a connected set for each $j \le i \le k$.

In order to prove (2), let us consider the set $W := \{z \in G^* / \, d_{G^*}(z, \gamma_0) \le 2\delta^*, \ d_{G^*}(z, T(\gamma_0)) \le 2\delta^*\} = G^* \cap \mathcal{V}_{2\delta^*}(\gamma_0) \cap \mathcal{V}_{2\delta^*}(T(\gamma_0))$. Since $\lim_{|z|\to\infty, z\in\gamma_0} d_G(z, T(\gamma_0)) = \lim_{|z|\to\infty, z\in\gamma_0} d_G(z, T^{-1}(\gamma_0)) = \infty$, the set W is compact so define $N_0 := \max\{d_{G^*}(z, \sigma \cap G^*) / \, z \in W\}$. Given any geodesic $g : [0, \ell] \to G^*$ joining a point in γ_0 with a point in $T(\gamma_0)$ (and contained in G^*) define

$$\alpha := \max\{s \in [0, \ell] / \, \eta(t) \in \mathcal{V}_{2\delta^*}(\gamma_0) \ \forall t \in [0, s]\}.$$

Let us consider the geodesic quadrilateral Q in G^* with sides $g =: [a_g b_g]$, $\sigma \cap G^* =: [a_\sigma b_\sigma]$, $[a_g a_\sigma] \subset \gamma_0$ and $[b_g b_\sigma] \subset T(\gamma_0)$. Since Q is $2\delta^*$-thin, $d_{G^*}(g(\alpha), \sigma \cap G^*) \le N := \max\{N_0, 2\delta^*\}$. (2) follows directly from this inequality. □

4. The main result.

Theorem 4.1. *Let G be a normalized periodic tessellation graph of $\mathbb{R}^2$. Then the following statements are equivalent:*

(1) *G is hyperbolic.*
(2) *G^* is hyperbolic and $\lim_{|z|\to\infty, z\in\gamma_0} d_G(z, T(z)) = \infty$ for some choice of the fundamental line γ_0.*
(3) *G^* is hyperbolic and $\lim_{|z|\to\infty, z\in\gamma_0} d_G(z, T(z)) = \infty$ for every choice of the fundamental line γ_0.*

Proof. Let us define $\gamma_u := \gamma_0 \cap U$, $G_u := G \cap U$ and $G_u^* := G^* \cap U$. Then, by symmetry and Lemma 3.10 applied to the geodesic line σ, it suffices to show the statement of Theorem 4.1 replacing γ_0, G and G^* by γ_u, G_u and G_u^*, respectively.

Let us prove the implication (1) ⇒ (3). Assume that G_u^* is not hyperbolic for some choice of the fundamental ray γ_u. Since γ_u and $T(\gamma_u)$ are geodesic rays, G_u^* is an isometric subgraph of G_u. Hence, Lemma 3.6 gives that G_u is not hyperbolic.

Assume now that for some choice of the fundamental ray γ_u, there exist a constant c_0 and a sequence $\{z_n\} \subset \gamma_u$ with $\lim_{n\to\infty} |z_n| = \infty$ and $d_G(z_n, T(z_n)) \le c_0$ for every n. Since $\cup_{n\in\mathbb{Z}} T^n(G_u^*) = U$, if $z_n = (x_n, y_n)$, then the sequence $\{y_n\}$ goes to infinity.

Let σ_n be a geodesic in G_u joining z_n and $T(z_n)$, and given $m \in \mathbb{N}$ let σ_n^m be the continuous curve in G_u joining z_n and $T^m(z_n)$ given by $\sigma_n^m := \cup_{j=0}^{m-1} T^j(\sigma_n)$. Next, it will be shown that σ_n^m, with its arc-length parametrization, is a $(c_0/k_0, 2k_0)$-quasigeodesic (recall that $T(x, y) = (x + k_0, y)$ and then $c_0 \ge d_G(z_n, T(z_n)) \ge d_{\mathbb{R}^2}(z_n, T(z_n)) = k_0$ and $c_0/k_0 \ge 1$). If $s < t$, directly, $d_G(\sigma_n^m(t), \sigma_n^m(s)) \le L(\sigma_n^m|_{[s,t]}) = t - s$. If $\sigma_n^m(s) \in T^j(\sigma_n)$ and $\sigma_n^m(t) \in T^{j+r}(\sigma_n)$, with $r \ge 0$, then

$$t - s \le (r + 1)L(\sigma_n) = (r + 1)d_{G_u}(z_n, T(z_n)) \le (r + 1)c_0,$$

$$d_{G_u}(\sigma_n^m(t), \sigma_n^m(s)) \ge d_{\mathbb{R}^2}(\sigma_n^m(t), \sigma_n^m(s)) \ge (r - 1)k_0 = (r + 1)c_0 \frac{k_0}{c_0} - 2k_0 \ge \frac{k_0}{c_0}(t - s) - 2k_0,$$

thus concluding σ_n^m is a $(c_0/k_0, 2k_0)$-quasigeodesic (for every n, m). If γ_u^n is the subcurve of γ_u joining z_1 and z_n, then let us choose a natural number $m = m(n)$ with $d_{\mathbb{R}^2}(\gamma_u^n, T^m(\gamma_u^n)) \ge n$. Hence, $Q_n := \{\gamma_u^n, \sigma_n^m, T^m(\gamma_u^n), \sigma_1^m\}$ is a $(c_0/k_0, 2k_0)$-quasigeodesic quadrilateral.

Seeking for a contradiction let us assume that G_u is hyperbolic. Let Q_n' be a geodesic quadrilateral in G_u with the same vertices than Q_n. By Theorem 2.2, the Hausdorff distance between a quasigeodesic side in Q_n and its corresponding geodesic side in Q_n' is less or equal than a constant $H = H(\delta(G_u), c_0/k_0, 2k_0)$. Let us show now that Q_n is $(2\delta(G_u) + 2H)$-thin. If p belongs to a side of Q_n, then there exists a point p' in its corresponding geodesic side in Q_n' at distance from p less or equal than H; since Q_n' is a geodesic quadrilateral, there exists a point q' in the union of the other three geodesic sides in Q_n' at distance from p' less or equal than $2\delta(G_u)$; then, there exists a point q in the union of the corresponding three quasigeodesic sides in Q_n at distance from q' less or equal than H, and $d_G(p, q) \le 2\delta(G_u) + 2H$. Hence, Q_n is $(2\delta(G_u) + 2H)$-thin.

Consider a point $p_n := (a_n, b_n) \in \gamma_u^n$ with $b_n = (y_n + y_1)/2$. Since $d_{\mathbb{R}^2}(z_n, T(z_n)) \le d_G(z_n, T(z_n)) \le c_0$,

$$d_{G_u}(p_n, \sigma_n^m \cup \sigma_1^m) \ge d_{\mathbb{R}^2}(p_n, \sigma_n^m \cup \sigma_1^m) \ge \frac{1}{2}(y_n - y_1) - c_0,$$
$$d_{G_u}(p_n, T^m(\gamma_0^n)) \ge d_{\mathbb{R}^2}(p_n, T^m(\gamma_0^n)) \ge n.$$

And therefore

$$\min\left\{\frac{1}{2}(y_n - y_1) - c_0,\ n\right\} \le 2\delta(G_u) + 2H,$$

for every n. This is a contradiction since $\{y_n\}$ goes to infinity, thus obtaining that G_u is not hyperbolic.

The implication (3) ⇒ (2) is direct.

Let us prove the implication (2) ⇒ (1). Define $\delta^* := \delta(G_u^*)$. Let us consider any geodesic triangle $\mathcal{T} = \{x_1, x_2, x_3\}$ with $x_i \in T^{j_i}(G_u^*)$ and $j_1 \le j_2 \le j_3$.

Since the constant M in Lemma 3.11 just depends on G^* and δ^*, one can assume that $[x_1x_2] \subset \cup_{j=j_1}^{j_2} T^j(G_u^*)$, $[x_2x_3] \subset \cup_{j=j_2}^{j_3} T^j(G_u^*)$, $[x_1x_3] \subset \cup_{j=j_1}^{j_3} T^j(G_u^*)$, and that $[x_1x_2] \cap T^i(G_u^*)$, $[x_2x_3] \cap T^i(G_u^*)$, $[x_1x_3] \cap T^i(G_u^*)$ are either the empty set or a connected set for each i.

Applying at most four times Lemma 3.10, one obtains that if $b - a \le 4$, then $\cup_{j=a}^{b} T^j(G_u^*)$ is δ_0-hyperbolic, with $\delta_0 = (120)^4\delta^*$.

By symmetry, it suffices to deal with the following cases:

(*a*) $j_2 - j_1 \le 2$ and $j_3 - j_2 \le 2$.
(*b*) $j_2 - j_1 \le 2$ and $j_3 - j_2 \ge 3$.
(*c*) $j_2 - j_1 \ge 3$ and $j_3 - j_2 \ge 3$.

Case (*a*). Since $\mathcal{T} \subset \cup_{j=j_1}^{j_3} T^j(G_u^*)$, with $j_3 - j_1 \le 4$, $\mathcal{T}$ is δ_0-thin.

Case (*b*). Let y_1 be the endpoint of $[x_1x_3] \cap \big(\cup_{j=j_1}^{j_2+1} T^j(G_u^*)\big)$ with $y_1 \in T^{j_2+2}(\gamma_u)$, and let y_2 be the endpoint of $[x_2x_3] \cap \big(\cup_{j=j_1}^{j_2+1} T^j(G_u^*)\big)$ with $y_2 \in T^{j_2+2}(\gamma_u)$. Consider the geodesic quadrilateral $\mathcal{Q} = \{x_1, x_2, y_2, y_1\}$ in $\cup_{j=j_1}^{j_2+1} T^j(G_u^*)$.

Let us bound $d_G(y_1, y_2)$. By Lemma 3.11 and Remark 3.12, for each $j_2 < j < j_3$, there exists a constant N, which just depends on G^* and δ^*, and points $z_1^j \in [x_1x_3] \cap T^j(G_u^*)$, $z_2^j \in [x_2x_3] \cap T^j(G_u^*)$, such that $d_{T^j(G_u^*)}(z_1^j, \sigma \cap T^j(G_u^*)), d_{T^j(G_u^*)}(z_2^j, \sigma \cap T^j(G_u^*)) \le N$. Consider $z \in [x_1x_3] \cap T^j(G_u^*)$ and $w \in [x_2x_3] \cap T^j(G_u^*)$, with $j_2 + 2 \le j \le j_3 - 2$. By defining $\ell := L(\sigma \cap G_u^*)$, then $d_G(z, z_1^j) \le \max\{d_G(z_1^{j-1}, z_1^j), d_G(z_1^j, z_1^{j+1})\} \le 2N + 2\ell$ and $d_G(w, z_2^j) \le \max\{d_G(z_2^{j-1}, z_2^j), d_G(z_2^j, z_2^{j+1})\} \le 2N + 2\ell$; since $d_G(z_1^j, z_2^j) \le 2N + \ell$, one obtains $d_G(z, w) \le 6N + 5\ell$ and, in particular, $d_G(y_1, y_2) \le 6N + 5\ell$.

Since $\cup_{j=j_1}^{j_2+1} T^j(G_u^*)$ is δ_0-hyperbolic, $\mathcal{Q}$ is $2\delta_0$-thin; therefore, given any point p in any side of $\mathcal{Q}$, there exists a point q in another side of $\mathcal{Q}$ with $d_G(p, q) \le 2\delta_0$. Hence, if $p \in \mathcal{Q} \cap \mathcal{T}$, then there exists a point q' in another side of $\mathcal{T}$ with $d_G(p, q) \le 2\delta_0 + 6N + 5\ell$.

If $z \in [x_1x_3] \cap \big(\cup_{j=j_2+2}^{j_3-2} T^j(G_u^*) \big)$, it was shown above $d_G(z, [x_2x_3]) \le 6N + 5\ell$. The same argument gives that if $w \in [x_2x_3] \cap \big(\cup_{j=j_2+2}^{j_3-2} T^j(G_u^*) \big)$, then $d_G(w, [x_1x_3]) \le 6N + 5\ell$.

Let y_1' be the endpoint of $[x_1x_3] \cap \big(\cup_{j=j_3-1}^{j_3} T^j(G_u^*) \big)$ with $y_1' \in T^{j_3-1}(\gamma_u)$, and let y_2' be the endpoint of $[x_2x_3] \cap \big(\cup_{j=j_3-1}^{j_3} T^j(G_u^*) \big)$ with $y_2' \in T^{j_3-1}(\gamma_u)$. Consider the geodesic triangle $\mathcal{T}_3 = \{y_1', x_3, y_2'\}$ in $\cup_{j=j_3-1}^{j_3} T^j(G_u^*)$. Since $d_G(y_1', y_2') \le 6N + 5\ell$.

Since $\cup_{j=j_3-1}^{j_3} T^j(G_u^*)$ is δ_0-hyperbolic, $\mathcal{T}_3$ is δ_0-thin; therefore, given any point p in any side of $\mathcal{T}_3$, there exists a point q in another side of $\mathcal{T}_3$ with $d_G(p, q) \le \delta_0$. Hence, if $p \in \mathcal{T}_3 \cap \mathcal{T}$, then there exists a point q' in another side of $\mathcal{T}$ with $d_G(p, q) \le \delta_0 + 6N + 5\ell$.

Case (c). Let a_1 be the endpoint of $[x_1x_2] \cap \big(\cup_{j=j_2-1}^{j_2+1} T^j(G_u^*) \big)$ with $a_1 \in T^{j_2-1}(\gamma_u)$, and let a_3 be the endpoint of $[x_2x_3] \cap \big(\cup_{j=j_2-1}^{j_2+1} T^j(G_u^*) \big)$ with $a_3 \in T^{j_2+2}(\gamma_u)$. Let b_1 and b_3 be, respectively, the endpoints of $[x_1x_3] \cap \big(\cup_{j=j_2-1}^{j_2+1} T^j(G_u^*) \big)$ with $b_1 \in T^{j_2-1}(\gamma_u)$ and $b_3 \in T^{j_2+2}(\gamma_u)$. Consider the geodesic pentagon $\mathcal{P} = \{a_1, x_2, a_3, b_3, b_1\}$ in $\cup_{j=j_2-1}^{j_2+1} T^j(G_u^*)$.

The previous argument in the Case (b) gives that $d_G(a_1, b_1), d_G(a_3, b_3) \le 6N + 5\ell$.

Since $\cup_{j=j_2-1}^{j_2+1} T^j(G_u^*)$ is δ_0-hyperbolic, $\mathcal{P}$ is $3\delta_0$-thin; therefore, given any point p in any side of $\mathcal{P}$, there exists a point q in another side of $\mathcal{P}$ with $d_G(p, q) \le 3\delta_0$. Hence, if $p \in \mathcal{P} \cap \mathcal{T}$, then there exists a point q' in another side of $\mathcal{T}$ with $d_G(p, q) \le 3\delta_0 + 6N + 5\ell$.

Finally, let us deal with the other situations as in the Case (b).

Hence, $\delta(G_u) \le 3\delta_0 + 6N + 5\ell$. □

5. Acknowledgements.

This work was partly supported by a grant from Ministerio de Ciencia e Innovación (MTM 2009-07800), Spain, and a grant from CONACYT (CONACYT-UAG I0110/62/10), México.

References

[1] Alonso, J., Brady, T., Cooper, D., Delzant, T., Ferlini, V., Lustig, M., Mihalik, M., Shapiro, M., Short, H.: Notes on word hyperbolic groups, in: E. Ghys, A. Haefliger, A. Verjovsky (Eds.), Group Theory from a Geometrical Viewpoint, World Scientific, Singapore, 1992.

[2] Ghys, E., de la Harpe, P.: Sur les Groupes Hyperboliques d'après Mikhael Gromov, Progress in Mathematics 83, Birkhäuser Boston Inc., Boston, MA, 1990.

[3] Gromov, M.: Hyperbolic groups, in "Essays in group theory". Edited by S. M. Gersten, M. S. R. I. Publ. 8, Springer, 1987, 75-263.

[4] Bermudo, S., Rodríguez, J. M., Sigarreta, J. M.: Computing the hyperbolicity constant, Comput. Math. Appl., 62, 4592-4595 (2011).

[5] Bermudo, S., Rodríguez, J. M., Sigarreta, J. M., Tourís, E.: Hyperbolicity and complement of graphs, Appl. Math. Letters, 24, 1882-1887 (2011).

[6] Brinkmann, G., Koolen J. and Moulton ,V.: On the hyperbolicity of chordal graphs, Ann. Comb., 5, 61-69 (2001).

[7] Carballosa, W., Pestana, D., Rodríguez, J. M., Sigarreta, J. M.: Distortion of the hyperbolicity constant of a graph, Electr. J. Comb., 19, P67 (2012).

[8] Carballosa, W., Rodríguez, J. M., Sigarreta, J. M., Villeta, M.: Gromov hyperbolicity of line graphs, Electr. J. Comb., 18, P210 (2011).

[9] Jonckheere, E., Lohsoonthorn, P.: A hyperbolic geometry approach to multipath routing, Proceedings of the 10th Mediterranean Conference on Control and Automation (MED 2002), Lisbon, Portugal, July 2002, FA5-1.
[10] Jonckheere, E. A.: Contrôle du traffic sur les réseaux à géométrie hyperbolique–Vers une théorie géométrique de la sécurité l'acheminement de l'information, J. Europ. Syst. Autom., 8, 45-60 (2002).
[11] Jonckheere, E. A., Lohsoonthorn, P.: Geometry of network security, Amer. Control Conf., ACC, 111-151 (2004).
[12] Koolen, J. H., Moulton, V.: Hyperbolic Bridged Graphs, Europ. J. Comb., 23, 683-699 (2002).
[13] Michel, J., Rodríguez, J. M., Sigarreta, J. M., Villeta, M.: Hyperbolicity and parameters of graphs, Ars Comb., Volume C, 43-63 (2011).
[14] Portilla, A., Rodríguez, J. M., Tourís, E.: Gromov hyperbolicity through decomposition of metric spaces II, J. Geom. Anal., 14, 123-149 (2004).
[15] Rodríguez, J. M., Sigarreta, J. M., Vilaire, J.-M., Villeta, M.: On the hyperbolicity constant in graphs, Discr. Math., 311, 211-219 (2011).
[16] Rodríguez, J. M., Tourís, E., Gromov hyperbolicity of Riemann surfaces, Acta Math. Sinica, 23, 209-228 (2007).
[17] Tourís, E.: Graphs and Gromov hyperbolicity of non-constant negatively curved surfaces, J. Math. Anal. Appl., 380, 865-881 (2011).
[18] Wu, Y., Zhang, C.: Chordality and hyperbolicity of a graph, Electr. J. Comb., 18, P43 (2011).
[19] Oshika, K.: Discrete groups, AMS Bookstore, 2002.
[20] Shavitt, Y., Tankel, T.: On internet embedding in hyperbolic spaces for overlay construction and distance estimation, INFOCOM 2004.
[21] Chepoi, V., Estellon, B.: Packing and covering δ-hyperbolic spaces by balls, APPROX-RANDOM, 59-73 (2007).
[22] Eppstein, D.: Squarepants in a tree: sum of subtree clustering and hyperbolic pants decomposition, SODA 2007, 29-38.
[23] Gavoille, C., Ly, O.: Distance labeling in hyperbolic graphs, ISAAC, Volume LNCS 3827, 2005, 1071-1079.
[24] Krauthgamer, R., Lee, J. R.: Algorithms on negatively curved spaces, FOCS, 2006.
[25] Bowditch, B. H.: Notes on Gromov's hyperbolicity criterion for path-metric spaces. Group theory from a geometrical viewpoint, Trieste, 1990 (ed. E. Ghys, A. Haefliger and A. Verjovsky; World Scientific, River Edge NJ, 1991, 64-167.
[26] Chen, B., Yau, S.-T., Yeh, Y.-N.: Graph homotopy and Graham homotopy, Discrete Math., 241, 153-170 (2001).